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Gravity in One Dimension: Diffusion in Acceleration

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The one-dimensional gravitational system consists of N parallel sheets of constant mass density. The sheets move perpendicular to their surface solely under their mutual gravitational attraction. When a pair has an encounter, they simply pass through each other. In this paper I consider the motion of a single sheet in an equilibrium ensemble. Under the assumption that the times separating encounters are random, I show that the acceleration and velocity (A, V) of a labeled sheet form a Markovian pair. Further, I prove that, in the limit of large N, (1) the (A, V) process is deterministic, (2) the (A, V) process obeys Vlasov dynamics, and (3) that scaled fluctuations in (A, V) comprise a diffusion which obeys a generalized Ornstein–Uhlenbeck process with time-dependent drift and diffusion tensors.

KEY WORDS: Stochastic process; diffusion; gravity; Vlasov equation.

1. INTRODUCTION

A cornerstone of nonequilibrium statistical physics is Brownian motion, the behavior of a massive particle in a homogeneous fluid.[1] The intuitive picture of the particle's motion is that it experiences random collisions with the fluid molecules. Consequently, the velocity of the heavy particle is a random (Markov) process. Since the mass of the Brownian particle is much greater than that of the molecules, the random change in its velocity due to a single collision is small. Therefore the velocity is well represented by a diffusion and its probability density function (pdf) satisfies a Fokker–Planck (FP) equation.[2] Under certain limiting circumstances this picture can be made rigorous. In one dimension, if the fluid is an ideal gas, Holley[3] has proven that, for a particular scaling, the velocity of the heavy particle obeys the Ornstein–Uhlenbeck (OU) process. Durr *et al.*[4] have

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extended these methods to three dimensions, while Miller and Stein[5] showed that the motion can be modeled by an alternative stochastic process which is also successful for nonthermal velocities of the heavy particle.

A central question in astrophysics concerns the stochastic nature of the dynamics of the elements of large systems which interact via gravitational forces (e.g., a molecule in an interstellar gas cloud, a star in a galaxy or globular cluster, or, perhaps, even a galaxy in an extremely large galactic cluster). It is well known that, in the limit of a large number of elements N of proportionally decreasing mass m, the system can be modeled by a continuous fluid in its μ space ($\mathbf{X}$, $\mathbf{V}$: $\mathbf{X}$ = position, $\mathbf{V}$ = velocity). The fluid density obeys the Vlasov (or Vlasov–Poisson) equation, sometimes referred to as the collisionless Boltzmann equation, and the Lagrangian dynamics of a given fluid element is deterministic.[6] Thus this picture cannot model the stochastic nature of individual elements (or particles).

In his classic work on stellar dynamics, Chandrasekhar modeled the motion of a star by assuming that two types of forces were acting: a smooth force produced by distant stars which arises from the negative gradient of the gravitational potential produced by the smooth average density of matter, i.e., the usual field in the Vlasov–Poisson equation, and an additional term representing the *lumpiness* in the true force as a consequence of near misses with other stars. The near misses were assumed to occur at random and provided the source of stochasticity in stellar systems.[7] Both astrophysicists and mathematicians have tried to improve on this intuitive picture. Heggie and Retterer[8] have transformed the problem of stellar motion to action–angle coordinates. They first set up the action–angle variables for the evolution of a star in a smooth mean field generated as above by the Vlasov density. To introduce stochasticity, they borrow an idea of Kaufman's[9] and model the canonical angle coordinates as a diffusion. For a very well-behaved class of interaction potentials (which does not include gravity in three or less dimensions), Braun and Hepp[10] have been able to prove the existence of a diffusion in position and velocity in the Vlasov limit. However, the long range of the gravitational force coupled with its singular short-range limit makes the task difficult for systems of real interest.

In this paper I consider the dynamics of a particle in a one-dimensional system in which mutual gravitational attraction provides the only forces acting on the system elements. The system can be visualized in three dimensions as N parallel sheets of constant mass density and infinite spatial extent. The motion of the sheets is restricted to the direction perpendicular to their surface. The evolution of the intersections of the sheets with a similarly directed line defines the system dynamics. Although this system is

not easily encountered in nature, it has been of some interest to astronomers and astrophysicists for a few decades. It has been employed to test a number of conjectures concerning galactic relaxation[11] and it is believed by some to model the motion of stars perpendicular to the plane of a highly flattened galaxy.[12]

Some of the central equilibrium properties of the one-dimensional gravitating system (OGS) were derived by Rybicki in 1971.[13] In particular, he found explicit expressions for the canonical partition function and singlet distribution functions in μ space in the canonical and microcanonical ensembles in terms of the system population N, mass M, and energy E. (There are no exact expressions for correlation functions, but there are some estimates.[14]) In the limit $N \to \infty$, with the total mass $Nm = M$ held constant, Rybicki proved that the μ-space distribution is proportional to $\operatorname{sech}^2(X) \exp(-V^2)$ in suitable units (see below). As expected, since there is no neutralizing background field as in the case of a plasma, the system is spatially inhomogeneous. The local potential induced by the equilibrium distribution of mass is proportional to $\ln[\cosh(X)]$.

Between encounters, the motion of a particle in the OGS is easy to describe: It simply undergoes a uniform acceleration due to the constant gravitational fields contributed by the remaining particles (sheets). Since the only forces acting are gravitational, the sheets simply pass through each other at an encounter. If it is assumed that all particles have the same mass (each sheet has the same mass density), then the particles exchange accelerations during an encounter. Thus, when viewed as a function of time, the position X of a given sheet and the velocity V are continuous functions, whereas the acceleration A is a sequence of discontinuous, equally spaced steps. Due to energy conservation, the accessible phase space is compact and a typical particle oscillates back and forth in the system. From dimensional considerations alone, the average period is on the order of $(G\rho)^{-1/2}$, where ρ is a typical system density. It is customary to define the characteristic time $t_c = (G\rho_0/\pi)^{-1/2}$, where ρ_0 is the equilibrium mass density evaluated at the origin and G is the universal gravitational constant.

In this paper I argue that, in the Vlasov limit, the acceleration and velocity of a labeled particle are deterministic and their scaled fluctuations are a diffusion. The diffusion ansatz works like this:

(1) If N is large, to a first approximation the total field a given particle experiences is roughly the mean field resulting from the equilibrium mass distribution.

(2) The true (instantaneous) acceleration of a system member depends on its sequence of encounter times.

(3) A typical particle experiences about N encounters in a time t_c.

The change in acceleration in a given small time interval depends solely on whether or not it has an encounter during that interval. Memory effects arise from reencounters with the same particle.[3,4] However, the mean time between these events is of the order of t_c, during which time *each particle* will experience about N encounters. Therefore, as N becomes large, it is unlikely that previous encounters with the same sheet will be "remembered."

(4) It is tempting to assume that the acceleration process alone is Markov. This is not possible, because both A and V are required to recover the Vlasov orbit in the limit $N \to \infty$. The pair A, V is the minimum set necessary to obtain the deterministic orbit in the Vlasov limit. Therefore I assume that, for large N, the (A, V) process is Markovian. The source of stochasticity is the random time for the next encounter, which is computed by conditioning on (A, V) with the equilibrium ensemble.

(5) As N becomes large, the size of the discontinuous jumps in A vanishes as $1/N$ and the encounter rate increases as N. Therefore, in the limit $N \to \infty$, the (A, V) process is a diffusion.

The central purpose of this paper is to analyze the consequences of the conjecture that the (A, V) process is Markov analytically. In what follows I will first develop the machinery for computing the transition probability for the state of the labeled particle. By taking the limit of short times, an exact difference equation governing the evolution of P, the pdf in (A, V) space, will be derived from the Chapman–Kolmogorov equation for the point process. I will then demonstrate that, as $N \to \infty$, P satisfies a first-order partial differential equation, proving that the A, V process is deterministic (no diffusion term). Fluctuations in (A, V) will then be scaled by $\sqrt{N}$ and I will show that, as $N \to \infty$, the process they define is also Markovian, but a diffusion, with one nonvanishing second moment. A Fokker–Planck equation governing the pdf for the scaled fluctuations will be derived and its features will be discussed. In a later paper Willard Maier and I will use actual dynamical simulations to demonstrate how well it works. Equivalently, it is possible to show that all powers but the first of the infinitesimal increments of A and V vanish as do all powers above the second for the fluctuations, leading to the identical formulation and conclusions.[2,5]

2. THE SETUP

Let m be the mass density of each sheet (equivalently, the mass of each particle). The gravitational field contributed by a single sheet is $2\pi mG$

directed toward the sheet. Since the field experienced by a particle is simply the sum of the constant fields produced by all the other particles, the acceleration of the jth particle from the left is proportional to the difference between the mass on its right and the mass on its left:

$$A_j = 2\pi Gm[N + 1 - 2j] \tag{1}$$

Because two masses simply pass through each other at an encounter, the position and velocity of each particle are continuous functions of the time, and the particles simply exchange their accelerations. Thus, except for questions concerning labeling, the equal-mass system considered here is isomorphic to one in which the particles collide elastically on contact.

To evaluate the probability of an encounter in a small time interval, I will use the canonical ensemble with the understanding that, in the limit of large N, the results are indistinguishable from those of the microcanonical ensemble. The central properties of each ensemble were carefully worked out by Rybicki in 1971.[13] Here I just borrow what I need.

The potential energy of a pair of sheets located at X_i and X_j is $2\pi Gm^2 |X_i - X_j|$. Thus the probability distribution in phase space is given by

$$\mathscr{P}_N(\mathbf{X}, \mathbf{P}) = \delta\left(\sum P_j\right) \delta\left(\sum X_j\right) \exp(-\beta E)/Z \tag{2}$$

where E is the total system energy,

$$E = \sum (P_j^2/2m) + 2\pi Gm^2 \sum_{i<j} |X_i - X_j| \tag{3}$$

Z is the partition function, $\beta = 1/kT$, and $(\mathbf{X}, \mathbf{P})$ signifies a point $(X_1, \ldots, P_N)$ in the $2N$-dimensional phase space. The delta functions ensure that the total momentum is zero (zero drift) and that the center of mass is fixed at the origin.

The potential energy in (3) is expressed as a sum over pairs of particles. It is much more useful to represent it as a sum over nearest neighbor pairs in the ordered configuration. This can be accomplished by noting that if the separations of all but one of the nearest neighbor pairs are fixed, then the work required to separate this pair by the distance U is $2\pi Gm^2 j(N-j)U$, where there are $j-1$ sheets to the left of the given pair. Thus the total potential energy of the N-particle system may be expressed as

$$\mathrm{PE} = 2\pi Gm^2 \sum_{1 \leqslant j \leqslant N-1} j(N-j)(Y_{j+1} - Y_j) \tag{4}$$

where, in (4), the $\{Y_j\}$ are selected from the $\{X_j\}$ by ordering them and relabeling so that $Y_{j+1} \geqslant Y_j$. We will call these coordinates the ordered configuration.

The previous result (4) suggests that, by introducing appropriate coordinates, we can express the phase-space pdf (2) as a product of independent functions of the separations between nearest neighbors. Let

$$U_j = Y_{j+1} - Y_j \geqslant 0; \quad j = 1,..., N-1; \qquad U_N = \sum Y_j \tag{5}$$

With a little algebra it is easily shown that the transformation preserves volume elements in the N-dimensional configuration space[13] ($|\text{Jacobian}| = 1$). In terms of the new coordinates the phase-space pdf, $\mathscr{P}_N(\mathbf{U}, \mathbf{P})$, takes the form of a product of exponential distributions,

$$\mathscr{P}_N(\mathbf{U}, \mathbf{P}) = (1/Z)\,\delta\left(\sum P_j\right)\delta(U_N)\exp(-\beta \cdot \mathrm{KE}) \times \prod_{1 \leqslant j \leqslant N-1} \Lambda_j \exp(-\Lambda_j U_j)\,\Theta(U_j) \tag{6}$$

where $\Theta(U)$ is the usual step function,

$$\mathrm{KE} = \sum P_j^2/2m \tag{7}$$

is the total kinetic energy,

$$\Lambda_j = m^2\beta 2\pi G j(N-j) = \Lambda(A_j) \tag{8}$$

and the temperature is related to the mean energy by

$$\beta\langle E\rangle = (3/2)(N-1) \tag{9}$$

Equation (9) is easily obtained by applying $\langle E\rangle = -\partial(\ln Z)/\partial\beta$ to (6) and reflects the fact that momentum conservation removes one degree of freedom.[13] It is apparent from (6) that the U_j are statistically independent. This is important for computing the probability of an encounter in a small time interval.

3. THE POINT PROCESS

The diffusion in velocity of an ordinary "Brownian" particle follows from the assumption that atomic collisions with the heavy particle occur at random times with random velocities. For the self-gravitating system with large population N, we focus on the acceleration and velocity, A and V, of a labeled particle. A changes as a result of encounters with other particles

in the system. Here we assume that encounters occur at random times. By conditioning on A and V, we use the canonical ensemble to compute the probability of an encounter with another particle in a short time interval. With this assumption, the evolution of the pair A, V is modeled by a Markov process. For finite N the changes in A are discontinuous, so we start with a point (jump) process, rather than a diffusion.[2] By taking the limit of large N, with $Nm = M$ held constant as above, we will show that (1) the A, V process is deterministic and (2) scaled fluctuations of A and V about their deterministic limits define a generalized Ornstein–Uhlenbeck process with time-dependent drift and diffusion tensors.[15,16]

3.1. Evolution of Probability

Let $P(A, V, t)$ be the probability distribution for the labeled particle. For finite N, A takes on discrete values [see (1)] and V is continuous. P obeys the Chapman–Kolmogorov equation[1,2,15,16] for the "conservation" of probability in which TP is the probability density of a transition from A', V' to A, $(V, V+dV)$ in the short time Δt:

$$P(A, V, t+\Delta t) = \int dV' \sum_{A'} TP(A, V; A', V')\, P(A', V', t) \tag{10}$$

The change in A due to an encounter is $\Delta A = \pm 4\pi G/N = \pm \Delta$, depending on whether the approaching particle is coming from the left or right. Thus, in the limit of small Δt, TP is the sum of the following three contributions:

$$\begin{aligned} TP = {} & \delta(V - V' - A'\,\Delta t)(\delta_{A,A'}\{1 - [p_R(A) + p_L(A)]\,\Delta t\} \\ & + \delta_{A,A'-\Delta}\, p_R(A')\,\Delta t + \delta_{A,A'+\Delta}\, p_L(A')\,\Delta t) \end{aligned} \tag{11}$$

respectively.

The crossing rates for encounters from the right and left, p_R and p_L, depend explicitly on A and V. We require the canonical phase space pdf to determine their functional dependence. Assume that the labeled particle is in the jth "slot" at the time t, i.e., that its acceleration is $A = A_j$. The condition for an encounter from the right [left] in the time interval Δt is $(V_j - V_{j+1})\,\Delta t > U_j - [(V_{j-1} - V_j)\,\Delta t > U_{j-1}]$. Using the canonical distribution, the conditional probability of an encounter from the right (left) through first order in Δt is

$$\begin{aligned} p_R(A)\,\Delta t &= A_j \langle (V - V_{j+1})\,\Theta(V - V_{j+1}) \,|\, V_j = V \rangle\,\Delta t \\ &\equiv (1/c)\, A_j f(Vc)\,\Delta t \end{aligned} \tag{12}$$

$$\begin{aligned} p_L(A)\,\Delta t &= A_{j-1} \langle (V_{j-1} - V)\,\Theta(V_{j-1} - V) \,|\, V_j = V \rangle\,\Delta t \\ &\equiv (1/c)\, A_{j-1} f(-Vc)\,\Delta t \end{aligned} \tag{13}$$

where, as usual, the angle brackets denote the ensemble average subject to the condition following the vertical bar, $c = (m\beta/2)^{1/2}$, and, to leading order in $1/N$,

$$f(x) = (1/2)[x + \pi^{-1/2} \exp(-x^2) + x \operatorname{erf}(x)] \tag{14}$$

By taking the limit of small Δt in (10), we obtain the following hybrid differential–difference equation for the evolution of $P(A, V, t)$:

$$\begin{aligned} &\partial P/\partial t + A\, \partial P/\partial V - (\Lambda_j/c) f(cV)[P(A+\Delta, V) - P(A, V)] \\ &\quad - (\Lambda_{j-1}/c) f(-cV)[P(A-\Delta, V) - P(A, V)] \\ &\quad + [(\Lambda_j - \Lambda_{j-1}) f(cV)\, P(A+\Delta, V)] \\ &\quad + [(\Lambda_{j-1} - \Lambda_j) f(-cV)\, P(A-\Delta, V)] = 0 \end{aligned} \tag{15}$$

where Λ_j and Λ_{j-1} are to be regarded as functions of the acceleration by solving for j in (1) and substituting in (8).

3.2. Rybicki Units

Because the potential energy is homogeneous to first order in the coordinates, the equations of motion can be reduced to dimensionless form. Moreover, all explicit system-dependent parameters, such as the total mass and energy, can be scaled away. Convenient units were introduced by Rybicki as follows:

$$\begin{aligned} \text{unit of length } L &= 2E/3\pi GM^2 \\ \text{unit of velocity} &= (4E/3M)^{1/2} \\ \text{unit of time } T &= (1/\pi MG)(E/3M)^{1/2} \end{aligned}$$

In these units, A, ΔA, Λ, and V are given by

$$\begin{aligned} A \to a = A/2\pi MG, \qquad \Delta \to \delta = 2/N, \qquad \Lambda \to \lambda = (N/2)(1-a^2) \\ V \to v = cV, \qquad t \to \tau = t/T, \qquad c \to 1 \end{aligned} \tag{16}$$

where we have dropped terms of order $1/N$ in λ, and M is the total system mass. From (15) it follows that the evolution equation for $P(a, v, \tau)$ is

$$\begin{aligned} &\partial P/\partial \tau + a\, \partial P/\partial v - \lambda f(v)[P(a+\delta, v, \tau) - P(a, v, \tau)] \\ &\quad - \lambda f(-v)[P(a-\delta, v, \tau) - P(a, v, \tau)] \\ &\quad + 2a[f(v)\, P(a+\delta) - f(-v)\, P(a-\delta)] = 0 \end{aligned} \tag{17}$$

Note that the only explicit system parameter which remains in (17) is the population N, which occurs in λ and δ.

4. VLASOV LIMIT AND THE DIFFUSION PROCESS

In the limit where N becomes large, the size of the acceleration jumps vanishes and the Markov process becomes a diffusion.

4.1. Vlasov Limit

Physically the Vlasov limit describes a system comprised of a continuous gravitating fluid in μ space. It can be obtained from the discrete case by letting the population diverge while controlling the total mass and energy.[6] In the limit $N \to \infty$, with $M = Nm$ (total mass/area) and E (total energy) constrained,

$$\lambda \to \infty, \qquad \delta \to 0, \qquad \lambda\delta \to 1 - a^2 \tag{18}$$

If we expand our version of the Chapman–Kolmogorov equation (17) in powers of δ and apply the Vlasov limit (18), we obtain

$$\partial P/\partial\tau + a\,\partial P/\partial v + 2avP - (1 - a^2)v\,\partial P/\partial a = 0 \tag{19}$$

This is a Fokker–Planck equation with vanishing diffusion tensor. Thus, in the Vlasov limit, the a, v process is deterministic: There is no noise.[2,5] The solution of (19) is simply

$$P(a, v, \tau) = \delta(a - \bar{a})\,\delta(v - \bar{v}) \tag{20}$$

where $\bar{a}(\tau)$, $\bar{v}(\tau)$ represent the deterministic evolution of a and v. Substitution of (20) into (19) yields a contribution from $\delta'(a - \bar{a})$ and another from $\delta'(v - \bar{v})$. For a solution, the coefficient of each must vanish identically. Thus, $\bar{a}(\tau)$ and $\bar{v}(\tau)$ obey the following ordinary differential equations:

$$d\bar{a}/d\tau = -(1 - \bar{a}^2)\bar{v}, \qquad d\bar{v}/d\tau = \bar{a} \tag{21}$$

This evolution corresponds to single-particle motion in the potential $\varphi(x)$ produced by the average equilibrium density $\rho(x)$, where

$$\varphi(x) = \ln\cosh x, \qquad \rho(x) = (1/2)\,\mathrm{sech}^2 x \tag{22}$$

This can be seen by regarding $x = x(\tau)$ as the trajectory of the "particle" and substituting $-d\varphi/dx = -\tanh[x(\tau)]$ for $\bar{a}$ in (21). Consequently, we recover the usual Vlasov theory, in which the system evolution takes the form of an incompressible flow in $\mu(x, v)$ space, from the (a, v) Markov process.

4.2. Fluctuations

We have seen that, in the Vlasov limit, the width of $P(a, v, \tau)$ vanishes, along with fluctuations in (a, v). In contrast, regardless of the magnitude of the system population, they are always present in the discrete system. In order to model their behavior for asymptotically large N they need to be amplified.

Let ξ, η represent the scaled fluctuations in the acceleration and velocity about their respective deterministic limits,

$$\xi = \sqrt{N}(a - \bar{a}), \qquad \eta = \sqrt{N}(v - \bar{v}) \tag{23}$$

For finite N, (ξ, η) still refers to the specific labeled particle and contains equivalent information to (a, v). Since $\bar{a}(\tau)$ and $\bar{v}(\tau)$ are known functions of the time given by (21) with appropriate initial conditions, and, by hypothesis, (a, v) is Markovian, so also is the pair (ξ, η). Because $a(t)$ changes discontinuously at encounters, so does ξ. However, here the size of the jumps go as $1/\sqrt{N}$. In the Vlasov limit, the jumps become vanishingly small and (ξ, η) becomes a diffusion.

We may use the definitions (23) to directly convert the difference equation (17) to an equation for the evolution of $P(\xi, \eta, \tau)$, the pdf for (ξ, η), by directly substituting $P(a, v)\, da\, dv \to P(\xi, \eta)\, d\xi\, d\eta$ and transforming variables. For finite N the result is still awkward. However, in the Vlasov limit, we find with a little work that the evolution equation is the Fokker–Planck equation

$$\partial_\tau P = -\partial_\eta(\xi P) - \partial_\xi[F(\xi, \eta, \tau)P] + (1/2)\,\partial_\xi^2[D(\tau)P] \tag{24}$$

in which $D(\tau)$ is the time-dependent diffusion constant,[5,15,16]

$$D(\tau) = 2(1 - \bar{a}^2)[f(\bar{v}) + f(-v)] \tag{25}$$

and $F(\xi, \eta, \tau)$ is the space- and time-dependent drift,[5,15,16]

$$F(\xi, \eta, \tau) = 2\bar{a}\bar{v}\xi - (1 - \bar{a}^2)\eta \tag{26}$$

Thus, ξ, η comprise a well-defined continuous Markov process (diffusion) which characterizes the drift of a labeled particle away from ideal Vlasov behavior.

The particular Fokker–Planck equation (24) belongs to a well-known class, the generalized Ornstein–Uhlenbeck process, which has its origins in Brownian motion.[15,16] (Here, as above, we have in mind the physical version of Brownian motion which models the behavior of a massive particle in a fluid.) Compared with the ordinary OU process, the state space is two

dimensional and the drift and diffusion tensors are time dependent.[15,16] In this particular case the time dependence is periodic and depends on the Lagrangian orbit of a point in the Vlasov fluid.[17] It is known, and is readily demonstrated by direct substitution, that (24) has a bivariate Gaussian solution.[16]

The central results of this section are embodied in Eqs. (21) and (24). Here they were derived by directly analyzing the transition probability in the Vlasov limit. Alternatively, they could have been obtained using the more general methods of ref. 5 by carrying out a moment expansion of the Chapman–Kolmagorov equation and examining the behavior of the infinitesimal increments for the (a, v) and the $(\xi. \eta)$ processes.

4.3. Usefulness of the Model

Although (24) is elegant, much simpler than (17), and solvable, by taking the Vlasov limit we have *formally* lost contact with the discrete system we wish to model. A similar phenomenon occurs in Brownian motion, where, depending on the particular limit taken, the mass of the colliding particles vanishes, or the mass of the heavy particle diverges.[3,5] In the Vlasov fluid any acceleration $\bar{a}$ is possible, whereas in the original system only discrete values occur. Important questions concern the relationships between the deterministic (a, v) process and the (ξ, η) diffusion and an actual realization of the physical system.

As N becomes large, the motion of an individual sheet remains close to its Vlasov image $(\bar{a}, \bar{v})$ for long times. $\langle \xi^2 \rangle$ and $\langle \eta^2 \rangle$, the time-dependent variances of ξ and η, provide a measure of the growth of the deviation from ideal Vlasov behavior in time. Because the diffusion model (24) is smooth, it cannot resolve the graininess of the actual accelerations: In the short-time limit, the solution of (24) will simply collapse to (20). Conversely, after sufficiently long times, the values of (a, v) for the test particle will differ considerably from its image: For *differences* on the order of, say, $([\langle a^2 \rangle]^{1/2}, [\langle v^2 \rangle]^{1/2})$ it is no longer reasonable to identify the test particle with its initial condition, which will be forgotten. Since, in Ribycki units, these quantities are of order unity, this will occur when the variances of ξ and/or η are on the order of N. Thus, it is expected that an ensemble of systems chosen by sampling the equilibrium distribution while conditioning on the initial state of the test particle will be successfully modeled by the diffusion process for times such that $(1/N) < \langle \xi^2 \rangle \ll N$, $\langle \eta^2 \rangle \ll N$.

5. CONCLUSIONS

The central conclusions of this study are:

1. In contrast with a single massive particle in an ordinary gas, where the pair of variables (X, V), or simply V alone, is a diffusion, in the one-dimensional gravitating system *each* member of the system diffuses. However, here only the *pair* of variables (A, V) is Markovian, A alone is definitely not. In addition, in the gravitating system it is not necessary to rescale the time. This is due to the fact that the mean encounter rate for the system scales as N^2. In the limit of large N, the pair A, V is deterministic and follows the usual Vlasov mean-field theory.

2. In the OGD, V, X, ε (=energy of a labeled particle) are computable processes, but they are *not* Markovian. This result may have relevance in astrophysics, where it is frequently assumed that ε can be modeled by a Markov process in which the random force is produced by distant stars in a globular cluster or galaxy.[18]

3. In the Vlasov limit, the pair of scaled fluctuations ξ, η about the deterministic acceleration and velocity is Markovian, and is controlled either by an inhomogeneous nonstationary stochastic differential equation or a Fokker–Planck equation (24). The pair (ξ, η) obey a generalized Ornstein–Uhlenbeck process with time-dependent drift and diffusion tensors. Information concerning the statistics of the process is contained in the pdf governing $P(\xi, \eta, \tau)$, which is known to have an explicit bivariate Gaussian solution.

4. Associated with an ensemble of similarly prepared one-dimensional gravitating systems at any time is a Vlasov fluid with mass density in μ space proportional to the singlet particle distribution function. If the system population N is large, the orbit in A, V space will closely approximate the orbit of a mass point in the fluid. However, as time progresses, the distance between the true element and its Vlasov "image" will increase on a time scale on the order of the crossing time of the system by approximately

$$(1/\sqrt{N}) \int_0^\tau \langle |\xi(t')| \rangle \, d\tau' \tag{27}$$

5. It is expected that the diffusion model is robust and will correctly model the single-particle fluctuations while the predicted variance remains considerably less than the system population.

In a followup investigation we will provide solutions to both the deterministic and Fokker–Planck equations (20) and (24) and compare them

with dynamical simulations of the system which are now underway. One preliminary result is that the time-periodic behavior of $\bar{a}$ and $\bar{v}$ results in unusual focusing effects for energetic particles which are dramatically verified in the simulations.

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